Time-space trade-off for SAT

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$DTISP(t(n), s(n))$ ... problems solvable by deterministic algorithms running in time $t(n)$ & space $s(n)$.

Q: $SAT \in DTIME(n)$? Open.

Thm 1: $NTIME(n) \not\subseteq DTISP(n^c, n^d)$ where $c = 1.4$, $d = \frac{1}{50}$

technique based on proofs of Kannan '83–84
proved by Lipton–Viglas 1999

- We will need several lemmas to prove the theorem.

Thm (Nepomjaščij): $DTISP(t, s) \subseteq \Sigma_2\text{-}TIME(\sqrt{t \cdot s})$

Pf: let L be from $DTISP(t, s)$ recognized by TM M running in time $t(n)$ & space $s(n)$.

Σ_2-alg. for L: on input $\underline{x}$ of length $\underline{n}$

- guess nondeterministically configurations $c_1, c_2, \dots c_{\sqrt{t/s}}$ of M on x.
- verify universally $\forall i \in \{1, \dots \sqrt{t/s} - 1\}$ that M goes from c_i to c_{i+1} in $\sqrt{ts}$ steps.
- check deterministically that c_1 is initial config of M on x & $c_{\sqrt{t/s}}$ is accepting configuration.

if so ACCEPT.

$\to$ $c_1 \dots c_{\sqrt{t/s}}$ requires $\sqrt{t/s} \cdot O(s)$ bits to describe so the step takes $O(\sqrt{ts})$ time.

for given i, checking $c_i \xrightarrow[\sqrt{ts} \text{ steps of } M]{} c_{i+1}$ can be done in time $O(\sqrt{ts})$ (assuming $\sqrt{ts} \geq n$) ∎

(2)

<u>Lemma</u>: If $NTIME(n) \subseteq DTIME(n^c)$ for some $c > 1$

then $\forall \tau(n) \geq n$

$$\Sigma_2\text{-}TIME(\tau(n)) \subseteq NTIME((\tau(n))^c).$$

<u>Pf</u>: $\Sigma_2\text{-}TIME(\tau(n)) = \exists y_1 \in \{0,1\}^{\tau} \forall y_2 \in \{0,1\}^{\tau} R(x, y_1, y_2)$

where $R(x, y_1, y_2)$ is a predicate computable in time linear w.r.t. $|x| + |y_1| + |y_2|$

From the assumption $NTIME(\tau(n)) \subseteq DTIME(\tau(n)^c)$

so the computation $\forall y_2 \in \{0,1\}^{\tau(n)} R(x, y_1, y_2) = R'(x, y_1)$

can be replaced by some det. computation

in time $O(\tau(n)^c)$ on input (x, y_1).

$\Longrightarrow$ we get $\exists y_1 \in \{0,1\}^{\tau(n)} \underbrace{R'(x, y_1)}_{DTIME(\tau(n)^c)}$

$$\in NTIME(\tau(n)^c)$$ ▨

<u>Pf of Thm 1</u>: By contradiction

Fix any $\tau(n) \geq n^2$ & assume $NTIME(n) \subseteq DTISP(n^c, n^d)$

$$\left.\begin{aligned} DTISP(\tau, \tau^{d/c}) &\underset{\text{(Nepomjaščij)}}{\subseteq} \Sigma_2\text{-}TIME(\tau^{\frac{1}{2} + \frac{d}{2c}}) \quad (\geq n) \\ &\underset{\text{(Lemma)}}{\subseteq} NTIME(\tau^{c/2 + d/2}) \end{aligned}\right\} (*)$$

by padding on the assumption

$$NTIME(\tau(n)) \subseteq DTISP(\tau(n)^c, \tau(n)^d) \subseteq$$

$$\underset{\text{by } (*)}{\subseteq} NTIME(\tau^{c^2/2 + dc/2})$$

if $\frac{c^2}{2} + \frac{dc}{2} < 1$ this contradicts nondeterministic time hierarchy ▨

[Williams '13]

- limit of this technique is $c = 1.81\ldots$

- limit of this technique

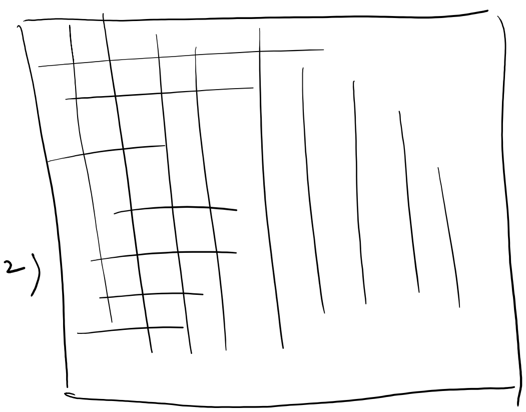
- SAT $\in$ NTIME$(n \lg^2 n)$ (Exc)
- NTIME(n) reduces to SAT of formulas of size $O(n \cdot \lg n)$.

... strengthening of Cook-Levin thm.
using "$t \lg t$" universal TM.

the normal Cook-Levin

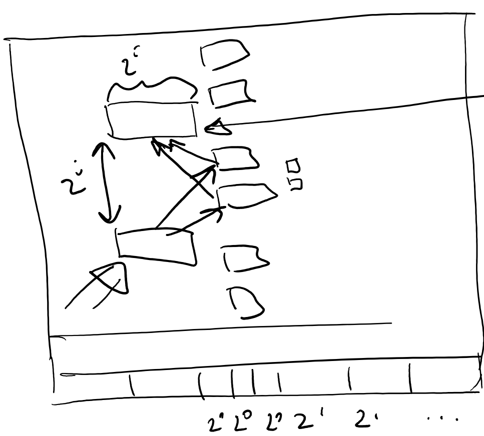
NTIME(n)

$\rightarrow$ ckt of size $O(n^2)$



more efficient using $t \lg t$ simulation strategy

$\rightarrow$ ckt of size $O(n \lg n)$



the block of size 2^i changes only every 2^i steps